学术圈知情人

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小议陈建龙的学术水平

An Algebraist

首先人物简介。根据公开资料，陈建龙是东南大学数学学院的二级教授。按理说应该有不错的学术造诣才能与这头衔匹配。

从他的教师个人主页提供的信息，陈教授好像没有接受过本科教育。当然，我对他在哪里接受过教育没有偏见，在那个年代嘛，寻常人能获得高等教育的机会实属难得。和他同龄的席南华，也没上过很好的本科，但席从没有避讳他毕业于怀化师范高等专科学校（现怀化学院）。

在这个鼓励学术争鸣的平台，同时也响应这篇文章的观点“把学术评价放在学术争鸣的平台上”，我来简要评价一下陈建

龙的学术水平。由于是学术评价而不是学术打假，自然会包含我个人的主观认识，失之偏颇的地方欢迎同行指出。

陈建龙从事的是基础数学的代数学研究，他的教师个人主页介绍其研究“涉及环论、模论、同调理论、代数K理论、环上线性代数、矩阵论及广义逆理论等领域”。听起来他的研究领域是非常广泛的，因为如果一个同行说他从事代数K理论的研究工作，我就已经听出他的研究内容是非常丰富的了。根据MathSciNet数据库的记录，他从业三十几年发表了三百多篇文章。这在代数学里算很高产的。从文章分类号看，陈教授的研究对象主要是关于环上的各种代数。

Classifications (11)

Classification	Publications	Citations
16 - Associative rings and algebras	202	1326
15 - Linear and multilinear algebra; matrix theory	89	669
47 - Operator theory	9	62
13 - Commutative algebra	7	15
46 - Functional analysis	6	18

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仔细看了一下他发表的论文标题。近十年来发表的一百三十多篇文章基本都是关于环上的广义逆理论。评价一个人的学术水平自然是要基于他的学术作品。我翻看了他近年来的一些文章。不看不知道，一看吓一跳，这研究简直不忍直视。

传统的实矩阵或复矩阵的广义逆研究已经出现好几本专著了，陈建龙考虑的是在环上的推广。虽然研究对象是更一般了，但基本定义还是类比原来的，研究框架也可以平行照搬。我觉得一个数学专业本科生在学过近世代数的基本语言之后，依葫芦画瓢也能够做出这样的简单推广。

我们一起来欣赏一下陈建龙的研究风格。他关于广义逆理论的研究全貌可以从论文标题

$\{\ldots, x_3, x_2, x_1\}$ **generalized inverses in** y **(with** z**)**

开始，其中 x, y, z可以替换成具体对象。注意，这里仅已知的具体对象经过各种组合就有数万种可能！下面通过举例来窥见一斑。

x_1的选择. Moore-Penrose 逆是最为人知一种广义逆，它由四个方程确定: $AXA = A$, $XAX = X, (AX)^* = AX, (XA)^* = XA$, 其中$A$是给定的矩阵。如果只要求$X$满足其中的一个或几个方程，也可以定义相应的广义逆。比如满足第一个方程的X叫作A的$\{1\}$-逆，同时满足前三个方程的X叫作A的$\{1,2,3\}$-逆。A的$\{1,2,3,4\}$-逆就是A的Moore-Penrose逆。理论上仅从这四个方程就能制造出$2^4 - 1 = 15$种逆。我们看看陈教授写了什么文章。

Wu, Cang; Chen, Jianlong; Greville type $\{1,2,3\}$*-generalized inverses for rectangular matrices, Filomat 38 (2024), no. 3, 803–809.*

Li, Tingting; Chen, Jianlong; Mosić, Dijana; Hua's identity for the $\{1,2\}$*-inverse, Bull. Iranian Math. Soc. 46 (2020), no. 2, 323–330.*

Xu, Sanzhang; Chen, Jianlong; *The Moore-Penrose inverse in rings with involution, Filomat 33 (2019), no. 18, 5791–5802.*

Wang, Long; Chen, Jian Long; A note on the Moore-Penrose inverse of a companion matrix, J. Southeast Univ. (English Ed.) 33 (2017), no. 1, 123–126.

Zhu, Huihui; Chen, Jianlong; Patrício, Pedro; The Moore-Penrose inverse of differences and products of projectors in a ring with involution, Turkish J. Math. 40 (2016), no. 6, 1316–1324.

Wang, Long; Chen, Jianlong; Mixed-type reverse-order laws of $(AB)^{(1,3)}, (AB)^{(1,2,3)}$ *and* $(AB)^{(1,3,4)}$*, Appl. Math. Comput. 222 (2013), 42–52.* [1]

如果换用其他类型的方程来定义广义逆呢[2]？勤劳的陈教授给我们展示了更多。

Wu, Cang; Chen, Jianlong; The $\{1,2,3,1^m\}$*-inverses: a generalization of core inverses for matrices, Appl. Math. Comput. 427 (2022), Paper No. 127149, 8 pp.*

Shi, Guiqi; Chen, Jianlong; On the set of (b,c)-invertible elements, Filomat 37 (2023), no. 9, 2743–2754.

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[1]这里的$(AB)^{(1,3)}$是指AB的$\{1,3\}$-逆。

[2]具体是什么方程这里就不做介绍了。

Zhu, Haiyang; Chen, Jianlong; Zhou, Yukun; On elements whose (b,c)-inverse is idempotent in a monoid, Filomat 36 (2022), no. 14, 4645–4653.

Wu, Cang; Chen, Jianlong; On (b,c)-inverses and (c,b)-inverses, Comm. Algebra 49 (2021), no. 10, 4313–4323.

Chen, Xiaofeng; Chen, Jianlong; The (b,c)-inverse in semigroups and rings with involution, Front. Math. China 15 (2020), no. 6, 1089–1104.

Xu, Sanzhang; Chen, Jianlong; Mosić, Dijana; The (j,m)-core inverse in rings with involution, Hacet. J. Math. Stat. 49 (2020), no. 5, 1676–1685. [3]

Mosić, Dijana; Zou, Honglin; Chen, Jianlong; On the (b,c)-inverse in rings, Filomat 32 (2018), no. 4, 1221–1231.

Ke, Yuanyuan; Cvetković-Ilić, Dragana S.; Chen, Jianlong; Višnjić, Jelena; New results on (b,c)-inverses, Linear Multilinear Algebra 66 (2018), no. 3, 447–458.

Ke, Yuanyuan; Wang, Zhou; Chen, Jianlong; The (b,c)-inverse for products and lower triangular matrices, J. Algebra Appl. 16 (2017), no. 12, 1750222, 17 pp.

Ke, Yuanyuan; Gao, Yuefeng; Chen, Jianlong; Representations of the (b,c)-inverses in rings with involution, Filomat 31 (2017), no. 9, 2867–2875.

Chen, J.; Ke, Y.; Mosić, D.; The reverse order law of the (b,c)-inverse in semigroups, Acta Math. Hungar. 151 (2017), no. 1, 181–198.

⋮ *the list is not exclusive.*

关于广义逆陈教授发表了这么多文章，我没有从文献中看到一种广义逆叫陈广义逆或建龙广义逆的。可能陈教授压根儿就没想去发明一组方程来定义新的广义逆，我猜大概原因是，已有的组合已经够他忙活一辈子了。

x_2的选择. 可以选 outer, left, right, one-sided 等。

Wu, Cang; Chen, Jianlong; A class of outer generalized inverses and a note on Drazin's finiteness conditions in rings and semigroups, Comm. Algebra 51 (2023), no. 12, 5167–5174.

Wu, Cang; Chen, Jianlong; Left core inverses in rings with involution, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 116 (2022), no. 2, Paper No. 67, 15 pp.

Chen, Xiaofeng; Chen, Jianlong; Right core inverses of a product and a companion matrix, Linear Multilinear Algebra 69 (2021), no. 12, 2245–2263.

[3]可以想象， 26个字母将不够用。保守估计这可以制造出一千万种逆，$2^{26}-1>>10^7$.

Ke, Yuanyuan; Chen, Jianlong; Stanimirović, Predrag; Ćirić, Miroslav; Characterizations and representations of outer inverse for matrices over a ring, Linear Multilinear Algebra 69 (2021), no. 1, 155–176.

Ke, Yuanyuan; Višnjić, Jelena; Chen, Jianlong; One-sided (b,c)-inverses in rings, Filomat 34 (2020), no. 3, 727–736.

Wang, Long; Castro-Gonzalez, Nieves; Chen, Jianlong; Characterizations of outer generalized inverses, Canad. Math. Bull. 60 (2017), no. 4, 861–871.

Chen, Jianlong; Zou, Honglin; Zhu, Huihui; Patrício, Pedro; The one-sided inverse along an element in semigroups and rings, Mediterr. J. Math. 14 (2017), no. 5, Paper No. 208, 17 pp.

⋮

x_3的选择. 可以选 weak, m-weak, strong, n-strong, weighted等。

Zhou, Yukun; Chen, Jianlong; Further characterizations of m-weak group inverses in a proper $*$-ring, Hacet. J. Math. Stat. 54 (2025), no. 6, 2295–2305.

Xu, Shuxian; Chen, Jianlong; Zhou, Yukun; Moore-Penrose m-weak group inverses in rings with involution, Filomat 39 (2025), no. 9, 2929–2940.

Li, Wende; Chen, Jianlong; Zhou, Yukun; Characterizations and properties of weak core inverses in rings with involution, Rocky Mountain J. Math. 54 (2024), no. 3, 793–807.

Li, Wende; Chen, Jianlong; Zhou, Yukun; Chen, Xiaofeng, Weighted generalized invertibility in two semigroups of a ring with involution, Filomat 38 (2024), no. 15, 5261–5274.

Wu, Cang; Chen, Jianlong; Strongly one-sided (b,c)-inverses in rings, Algebra Colloq. 31 (2024), no. 4, 549–558.

Zhou, Yukun; Chen, Jianlong; Jacobson's lemma and Cline's formula for weighted generalized inverses in a ring with involution, Filomat, 37 (2023), no. 16, 5313-5324.

Zhou, Yukun; Chen, Jianlong; Weak core inverses and pseudo core inverses in a ring with involution, Linear Multilinear Algebra 70 (2022), no. 21, 6876–6890.

Zou, Honglin; Chen, Jianlong; Zhu, Huihui; Wei, Yujie; Characterizations for the n-strong Drazin invertibility in a ring, J. Algebra Appl. 20 (2021), no. 8, Paper No. 2150141, 18 pp.

⋮

generalized inverses 还可以换成更具体的一些广义逆，比如 Drazin inverse, pseudo Drazin inverse, group inverse, core inverse 等，陈教授用了多篇文章将这些广义逆悉数照顾到。

y的选择. 可以选 semigroups, rings, $*$-semigroups, $*$-rings, a unitary ring, a monoid, a proper $*$-ring, Banach algebras等等。不胜枚举，读者可自行上MathSciNet或ZbMath查看。

z的选择. 可以留空或者选 involution. 加了involution, 论文数量又可以翻倍.

我注意到陈教授近十年的文章基本是和他的研究生共同署名。这些dirty work估计都是他的学生来做。用这样索然无味的素材，低级的研究水准来训练研究生，难道从来就没有为学生考虑过吗？经他指导的学生除了能轻松学到怎么炮制文章外还能学到什么？翻找一圈后，我好不容易找到了陈教授独著的文章

[1]. Chen, Jianlong; Algebraic theory of generalized inverses: algebraic basic knowledge and Moore-Penrose inverses, Nanjing Daxue Xuebao Shuxue Bannian Kan 37 (2020), no. 2, 106–212.

[2]. Chen, Jianlong; Algebraic theory of generalized inverses: group inverses and Drazin inverses, Nanjing Daxue Xuebao Shuxue Bannian Kan 38 (2021), no. 1, 1–113.

[3]. Chen, Jianlong; Algebraic theory of generalized inverses: core inverses and pseudo core inverses, Nanjing Daxue Xuebao Shuxue Bannian Kan 39 (2022), no. 1, 1–107.

乍一看，陈教授似乎还挺用功的，独著的文章能写这么长。原本我是想找独著文章去更客观地评价陈教授的研究水准。不过我只用了几分钟就看完这几百页内容，原因是，这三篇文章没有证明任何新结果，完全是将自己和他人的定理和证明罗列一遍。弄得我也搞不清楚《南京大学学报(数学半年刊)》是什么性质的刊物。有些该标注的地方没标注，比如文[1]第107页的The core-nilpotent decomposition, 这明显是已知的结果，作者却没有标注参考文献，这个分解可以从经典广义逆书籍 Adi Ben-Israel & Thomas Greville, Generalized Inverses: Theory and Applications, 2003年第2版的第169页找到。文[1]第142页关于Moore-Penrose逆的存在唯一性证明是所有广义逆书籍都收录的，为何陈教授要在文章中再记录

一次证明。难道陈教授是要写书吗？还真被我猜中了。2024年陈建龙和他的原博士生张小向教授出版了一本书

Chen, Jianlong; Zhang, Xiaoxiang; Algebraic theory of generalized inverses, Science Press, Beijing; Springer, Singapore, 2024, xi+322 pp.

不过令人费解的是这本书居然没有引用陈教授的文献[1, 2, 3]。想必陈教授早已知道文献[1, 2, 3]只是已有结果的搬运工。仔细一看，文献[1, 2, 3]按顺序放在一起，完全就是陈建龙和张小向的书了。弄不好，这里还可能会出现重复发表的学术不端问题。

陈教授的专业是基础数学，但他已发表的学术论文并没有体现这一点。主要是基本的审美完全缺失（在附录中我截取了文[2]的两个定理及证明让大家感受一下，类似场景在文[3]的第38页至45页，第55页至60页也都出现过）；其次是论文的署名顺序毫无规范。

我的总体观察是：陈建龙教授关于广义逆理论的研究在形式上完整，思想上空洞，并未围绕任何公认的重要问题展开。他在这方面的研究体现出典型的对象驱动(object-driven) 特征，而非问题驱动(problem-driven)研究。其直接动机是：可以炮制大量的文章。但这些技术性堆砌的文章乏味无趣。纵观陈教授近十年来的学术研究，他并未引入新的方法，也无揭示新的代数结构。他的学术论文可以被归类为低层级技术性扩展(low-level technical elaboration)。从学术评价角度，这种模式被视为以对象替换来实现发表(object-substitution driven publication)。

我和陈教授没有任何利益冲突。他可以接连获得国家自然科学基金资助，我确实挺好奇专家们是怎么评价他在项目申请书中所展示的研究内容。之所以要来评价一番，主要是不想让年轻人看到这么丑陋的数学。如果一个年轻人看到的代数学是陈教授所展示的那番模样，充斥着那么乏味无趣的内容，谁还会愿意去学代数，是吧？

附录

From (1)⇒(2), we know that $1_X - \gamma$ and $1_X - \delta$ are invertible and $\omega = (1_X - \gamma)^{-1}$, $t = (1_X - \delta)^{-1}$. So $f^{\#} = (1_X - \gamma)^{-1}\alpha\varphi^{\#}(1_X - \delta)^{-1}$.

Remark 4.2([49, Remark 1]) The method for proving (3)⇒(1) belongs to Hartwig (see [23]).

Corollary 4.3([28, Proposition 3]) If $a \in R$ has a group inverse $a^{\#}$ and $j \in J(R)$, then $a+j$ has a group inverse if and only if $(1-aa^{\#})j(1+a^{\#}j)^{-1}(1-a^{\#}a) = 0$.

In that case, $(a+j)^{\#} = (1-\gamma)^{-1}(1+a^{\#}j)^{-1}a^{\#}(1-\delta)^{-1}$, in which

$$\begin{aligned}\gamma &= (1+a^{\#}j)^{-1}(1-a^{\#}a)ja^{\#}(1+ja^{\#})^{-1},\\ \delta &= (1+a^{\#}j)^{-1}a^{\#}j(1-aa^{\#})(1+ja^{\#})^{-1}.\end{aligned}$$

Proof Since $j \in J(R)$, the element $1+a^{\#}j$ is invertible.

"⇐". By hypothesis, $\varepsilon = 0$, so $f = a+j$. Since $1-\gamma$ and $1-\delta$ are invertible, $a+j$ has a group inverse by Proposition 4.1. In that case, $(a+j)^{\#} = (1-\gamma)^{-1}(1+a^{\#}j)^{-1}a^{\#}(1-\delta)^{-1}$.

"⇒". Let $\tau = (a+j)^{\#}$. By [11, Lemma 14.1], $\tau \in \varepsilon\{1\}$, i.e., $\varepsilon\tau\varepsilon = \varepsilon$ and $\varepsilon(1-\tau\varepsilon) = 0$. Since $\varepsilon = (1-aa^{\#})j(1+a^{\#}j)^{-1}(1-a^{\#}a) \in J(R), 1-\tau\varepsilon$ is invertible and $\varepsilon = 0$.

5 The Group Inverse of the Sum of Two Group Invertible Elements

In this section, we assume that R is a Dedekind-finite ring. The group inverses of sum and difference of two group invertible elements are presented under different conditions.

Theorem 5.1([52, Theorem 3.1]) Let $a,\ b \in R^{\#}$ and $2 \in R^{-1}$. If $abb^{\#} = baa^{\#}$, then

(1) $a+b \in R^{\#}$ and $(a+b)^{\#} = a^{\#} + b^{\#} - \frac{1}{2}a^{\#}bb^{\#} - \frac{1}{2}bb^{\#}a^{\#} - \frac{1}{2}aa^{\#}b^{\#}$.

(2) $a-b \in R^{\#}$ and $(a-b)^{\#} = a^{\#} - b^{\#} - ba^{\#}a^{\#} + ab^{\#}b^{\#}$.

Proof Since $abb^{\#} = baa^{\#}$, we have $abb^{\#}(b^{\#}aa^{\#})abb^{\#} = ab^{\#}abb^{\#} = ab^{\#}baa^{\#} =$

$abb^{\#}$. So $b^{\#}aa^{\#}$ is an inner inverse of $abb^{\#}$. Then we have

$$\begin{aligned}&(1+b^{\#}aa^{\#}-b^{\#}abb^{\#})(1+abb^{\#}-b^{\#}abb^{\#})=1+abb^{\#}\\&-b^{\#}abb^{\#}+b^{\#}aa^{\#}+b^{\#}abb^{\#}-b^{\#}aa^{\#}b^{\#}abb^{\#}-b^{\#}abb^{\#}-b^{\#}babb^{\#}+b^{\#}ab^{\#}abb^{\#}\\=&1+abb^{\#}+b^{\#}aa^{\#}-b^{\#}aa^{\#}b^{\#}baa^{\#}-b^{\#}baa^{\#}-b^{\#}babb^{\#}+b^{\#}ab^{\#}baa^{\#}\\=&1+abb^{\#}+b^{\#}aa^{\#}-b^{\#}b^{\#}baa^{\#}b^{\#}baa^{\#}-b^{\#}baa^{\#}-b^{\#}babb^{\#}+b^{\#}baa^{\#}\\=&1+abb^{\#}+b^{\#}aa^{\#}-b^{\#}aa^{\#}-b^{\#}baa^{\#}-baa^{\#}+b^{\#}baa^{\#}=1.\end{aligned}$$

Since R is a Dedekind-finite ring, $(1+abb^{\#}-b^{\#}abb^{\#})(1+b^{\#}aa^{\#}-b^{\#}abb^{\#})=1$. From Theorem 2.7, we know that $abb^{\#}\in R^{\#}$. Since $b^{\#}(baa^{\#})^2=b^{\#}babb^{\#}=b^{\#}bbaa^{\#}=baa^{\#}$, we have

$$\begin{aligned}b^{\#}aa^{\#}&=(b^{\#})^2baa^{\#}=(b^{\#})^2(baa^{\#})^3[(baa^{\#})^{\#}]^2\\&=(baa^{\#})[(baa^{\#})^{\#}]^2=(baa^{\#})^{\#}.\end{aligned}$$

Similarly, $(baa^{\#})^{\#}=a^{\#}bb^{\#}$. So $a^{\#}bb^{\#}=b^{\#}aa^{\#}$. Furthermore, $aa^{\#}bb^{\#}=ab^{\#}aa^{\#}=abb^{\#}b^{\#}aa^{\#}=b^{\#}aa^{\#}abb^{\#}=b^{\#}abb^{\#}=b^{\#}baa^{\#}$. In addition,

$$bb^{\#}ab^{\#}=bb^{\#}abb^{\#}b^{\#}=bb^{\#}baa^{\#}b^{\#}=baa^{\#}b^{\#}=abb^{\#}b^{\#}=ab^{\#},$$

$$bb^{\#}a^{\#}b=bb^{\#}(a^{\#}bb^{\#})b=bb^{\#}b^{\#}aa^{\#}b=(b^{\#}aa^{\#})b=a^{\#}bb^{\#}b=a^{\#}b.$$

Similarly, $aa^{\#}ba^{\#}=ba^{\#}$ and $aa^{\#}b^{\#}a=b^{\#}a$. We know that $abaa^{\#}=abb^{\#}baa^{\#}=baa^{\#}abb^{\#}=babb^{\#}=bbaa^{\#}$. Hence $aba^{\#}=bba^{\#}$. Since $a^{\#}b^{\#}aa^{\#}=a^{\#}bb^{\#}b^{\#}aa^{\#}=b^{\#}aa^{\#}a^{\#}bb^{\#}=b^{\#}a^{\#}bb^{\#}=b^{\#}b^{\#}aa^{\#}$, $a^{\#}b^{\#}a=b^{\#}b^{\#}a$. Similarly, we can get $bab^{\#}=aab^{\#}$ and $b^{\#}a^{\#}b=a^{\#}a^{\#}b$. From the above discussion, we have

$$abb^{\#}=baa^{\#},\tag{5.10}$$

$$a^{\#}bb^{\#}=b^{\#}aa^{\#},\ aa^{\#}bb^{\#}=bb^{\#}aa^{\#},\tag{5.11}$$

$$aa^{\#}b^{\#}a=b^{\#}a,\ bb^{\#}a^{\#}b=a^{\#}b,\tag{5.12}$$

$$bab^{\#}=aab^{\#},\ aba^{\#}=bba^{\#},\tag{5.13}$$

$$a^{\#}b^{\#}a=b^{\#}b^{\#}a,\ a^{\#}a^{\#}b=b^{\#}a^{\#}b,\tag{5.14}$$

$$aa^{\#}ba^{\#}=ba^{\#},\ bb^{\#}ab^{\#}=ab^{\#}.\tag{5.15}$$

(1) Let $x=a^{\#}+b^{\#}-\frac{1}{2}a^{\#}bb^{\#}-\frac{1}{2}bb^{\#}a^{\#}-\frac{1}{2}aa^{\#}b^{\#}$. By equalities (5.10)-(5.12),

then

$$\begin{aligned}(a+b)x=&(a+b)(a^{\#}+b^{\#}-\frac{1}{2}a^{\#}bb^{\#}-\frac{1}{2}bb^{\#}a^{\#}-\frac{1}{2}aa^{\#}b^{\#})\\
=&aa^{\#}+ab^{\#}-\frac{1}{2}aa^{\#}bb^{\#}-\frac{1}{2}abb^{\#}a^{\#}\\
&-\frac{1}{2}ab^{\#}+ba^{\#}+bb^{\#}-\frac{1}{2}ba^{\#}bb^{\#}-\frac{1}{2}ba^{\#}-\frac{1}{2}baa^{\#}b^{\#}\\
=&a^{\#}a+ab^{\#}-\frac{1}{2}aa^{\#}bb^{\#}-\frac{1}{2}ba^{\#}-\frac{1}{2}ab^{\#}+ba^{\#}+bb^{\#}\\
&-\frac{1}{2}aa^{\#}bb^{\#}-\frac{1}{2}ba^{\#}-\frac{1}{2}ab^{\#}\\
=&aa^{\#}+bb^{\#}-aa^{\#}bb^{\#}.\end{aligned}$$

and

$$\begin{aligned}x(a+b)&=(a^{\#}+b^{\#}-\frac{1}{2}a^{\#}bb^{\#}-\frac{1}{2}bb^{\#}a^{\#}-\frac{1}{2}aa^{\#}b^{\#})(a+b)\\
&=a^{\#}a+b^{\#}a-\frac{1}{2}b^{\#}a-\frac{1}{2}aa^{\#}bb^{\#}-\frac{1}{2}b^{\#}a+b^{\#}b-\frac{1}{2}aa^{\#}bb^{\#}\\
&=aa^{\#}+bb^{\#}-aa^{\#}bb^{\#}.\end{aligned}$$

So $(a+b)x=x(a+b)$. By equalities (5.10) and (5.11), we have

$$\begin{aligned}x(a+b)x=&(aa^{\#}+bb^{\#}-aa^{\#}bb^{\#})(a^{\#}+b^{\#}-\frac{1}{2}a^{\#}bb^{\#}-\frac{1}{2}bb^{\#}a^{\#}-\frac{1}{2}aa^{\#}b^{\#})\\
=&a^{\#}+aa^{\#}b^{\#}-\frac{1}{2}a^{\#}bb^{\#}-\frac{1}{2}bb^{\#}a^{\#}-\frac{1}{2}aa^{\#}b^{\#}+bb^{\#}a^{\#}+b^{\#}-\frac{1}{2}a^{\#}bb^{\#}\\
&-\frac{1}{2}bb^{\#}a^{\#}-\frac{1}{2}aa^{\#}b^{\#}-bb^{\#}a^{\#}-aa^{\#}b^{\#}+\frac{1}{2}a^{\#}bb^{\#}+\frac{1}{2}bb^{\#}a^{\#}+\frac{1}{2}aa^{\#}b^{\#}\\
=&a^{\#}+b^{\#}-\frac{1}{2}a^{\#}bb^{\#}-\frac{1}{2}bb^{\#}a^{\#}-\frac{1}{2}aa^{\#}b^{\#}=x\end{aligned}$$

and

$$\begin{aligned}(a+b)x(a+b)&=(a+b)(aa^{\#}+bb^{\#}-aa^{\#}bb^{\#})\\
&=a+abb^{\#}-abb^{\#}+baa^{\#}+b-baa^{\#}bb^{\#}\\
&=a+b.\end{aligned}$$

So $(a+b)^{\#}=a^{\#}+b^{\#}-\frac{1}{2}a^{\#}bb^{\#}-\frac{1}{2}bb^{\#}a^{\#}-\frac{1}{2}aa^{\#}b^{\#}$.

(2) Let $y=a^{\#}-b^{\#}-ba^{\#}a^{\#}+ab^{\#}b^{\#}$. By equalities (5.12)-(5.14), then

$$\begin{aligned}(a-b)y&=(a-b)(a^{\#}-b^{\#}-ba^{\#}a^{\#}+ab^{\#}b^{\#})\\
&=aa^{\#}-ab^{\#}-aba^{\#}a^{\#}+aab^{\#}b^{\#}-ba^{\#}+bb^{\#}+bba^{\#}a^{\#}-bab^{\#}b^{\#}\\
&=aa^{\#}-ab^{\#}-bba^{\#}a^{\#}+aab^{\#}b^{\#}-ba^{\#}+bb^{\#}+bba^{\#}a^{\#}-aab^{\#}b^{\#}\\
&=aa^{\#}-ab^{\#}-ba^{\#}+bb^{\#}\end{aligned}$$

and

$$\begin{aligned}y(a-b) &= (a^\# - b^\# - ba^\#a^\# + ab^\#b^\#)(a-b)\\ &= a^\#a - b^\#a - ba^\#a^\#a + ab^\#b^\#a - a^\#b + b^\#b + ba^\#a^\#b - ab^\#b^\#b\\ &= a^\#a - b^\#a - ba^\# + b^\#a - a^\#b + b^\#b + a^\#b - ab^\#\\ &= aa^\# - ab^\# - ba^\# + bb^\#.\end{aligned}$$

So $(a-b)y = y(a-b)$. By equalities (5.11) and (5.15), we have

$$\begin{aligned}&ab^\#a^\# = a(b^\#aa^\#)a^\# = aa^\#bb^\#a^\#,\\ &ba^\#b^\# = b(a^\#bb^\#)b^\# = bb^\#aa^\#b^\#,\\ &ba^\#ba^\#a^\# = b(a^\#bb^\#)ba^\#a^\# = (bb^\#aa^\#)ba^\#a^\# = (aa^\#ba^\#)a^\# = ba^\#a^\#,\\ &ab^\#ab^\#b^\# = ab^\#(aa^\#a)b^\#b^\# = a(b^\#aa^\#)ab^\#b^\# = (aa^\#bb^\#)ab^\#b^\#\\ &= bb^\#aa^\#ab^\#b^\# = (bb^\#ab^\#)b^\# = ab^\#b^\#.\end{aligned}$$

Thus,

$$\begin{aligned}&y(a-b)y = (aa^\# - ab^\# - ba^\# + bb^\#)(a^\# - b^\# - ba^\#a^\# + ab^\#b^\#)\\ =&a^\# - aa^\#bb^\#a^\# - ba^\#a^\# + bb^\#a^\# - ba^\#a^\# + ba^\#a^\# + ba^\#a^\# - ba^\#a^\#\\ &- aa^\#b^\# + ab^\#b^\# + bb^\#aa^\#b^\# - b^\# + ab^\#b^\# - ab^\#b^\# - ab^\#b^\# + ab^\#b^\#\\ =&a^\# - b^\# - ba^\#a^\# + ab^\#b^\# = y.\end{aligned}$$

Since

$$ab^\#a = a(b^\#aa^\#)a = aa^\#bb^\#a,\ \ ba^\#b = b(a^\#bb^\#)b = bb^\#aa^\#b,$$

we have

$$\begin{aligned}(a-b)y(a-b) &= (aa^\# - ab^\# - ba^\# + bb^\#)(a-b)\\ &= a - ab^\#a - ba^\#a + bb^\#a - aa^\#b + ab^\#b + ba^\#b - b\\ &= a - aa^\#bb^\#a + bb^\#a - aa^\#b + bb^\#aa^\#b - b\\ &= a - bb^\#a + bb^\#a - aa^\#b + aa^\#b - b = a - b.\end{aligned}$$

Therefore, $(a-b)^\# = a^\# - b^\# - ba^\#a^\# + ab^\#b^\#$.

Next, we present the expressions of $(a+b)^\#$ and $(a-b)^\#$ under the condition $bb^\#a = aa^\#b$.

Theorem 5.2([52, Theorem 3.2]) Let $a,\ b \in R^\#$ and $2 \in R^{-1}$. If $bb^\#a = aa^\#b$, then

(1) $a+b\in R^{\#}$ and $(a+b)^{\#}=a^{\#}+b^{\#}-\frac{1}{2}bb^{\#}a^{\#}-\frac{1}{2}a^{\#}bb^{\#}-\frac{1}{2}b^{\#}aa^{\#}$.

(2) $a-b\in R^{\#}$ and $(a-b)^{\#}=a^{\#}-b^{\#}-a^{\#}a^{\#}b+b^{\#}b^{\#}a$.

Proof Since $bb^{\#}a(aa^{\#}b^{\#})bb^{\#}a=bb^{\#}ab^{\#}a=aa^{\#}bb^{\#}a=bb^{\#}a$, $aa^{\#}b^{\#}$ is an inner inverse of $bb^{\#}a$. Since $bb^{\#}a=aa^{\#}b$, we have

$$
\begin{aligned}
&(1+bb^{\#}a-bb^{\#}ab^{\#})(1+aa^{\#}b^{\#}-bb^{\#}ab^{\#})=1+bb^{\#}a-bb^{\#}ab^{\#}\\
&\quad+aa^{\#}b^{\#}+bb^{\#}ab^{\#}-bb^{\#}ab^{\#}aa^{\#}b^{\#}-bb^{\#}ab^{\#}-bb^{\#}abb^{\#}ab^{\#}+bb^{\#}ab^{\#}ab^{\#}\\
&=1+bb^{\#}a+aa^{\#}b^{\#}-aa^{\#}bb^{\#}aa^{\#}b^{\#}-bb^{\#}ab^{\#}-bb^{\#}abb^{\#}+aa^{\#}bb^{\#}ab^{\#}\\
&=1+bb^{\#}a+aa^{\#}b^{\#}-aa^{\#}b^{\#}-aa^{\#}bb^{\#}-aa^{\#}b+aa^{\#}bb^{\#}=1.
\end{aligned}
$$

Since R is a Dedekind-finite ring, $(1+aa^{\#}b^{\#}-bb^{\#}ab^{\#})(1+bb^{\#}a-bb^{\#}ab^{\#})=1$. So $1+bb^{\#}a-bb^{\#}ab^{\#}\in R^{-1}$. From Theorem 2.7, we know that $bb^{\#}a\in R^{\#}$. Since $bb^{\#}aa^{\#}a=bb^{\#}a=aa^{\#}b$, $(bb^{\#}a)^{\#}aa^{\#}=aa^{\#}b^{\#}$. Then we have

$$(bb^{\#}a)^{\#}aa^{\#}=(bb^{\#}a)^{\#}(bb^{\#}a)^{\#}bb^{\#}aaa^{\#}=(bb^{\#}a)^{\#}.$$

Hence $(bb^{\#}a)^{\#}=aa^{\#}b^{\#}$. Similarly, $(aa^{\#}b)^{\#}=bb^{\#}a^{\#}$. Therefore, $bb^{\#}a^{\#}=aa^{\#}b^{\#}$. We have

$$aa^{\#}bb^{\#}=bb^{\#}ab^{\#}=(bb^{\#}a)(aa^{\#}b^{\#})=(aa^{\#}b^{\#})(bb^{\#}a)=bb^{\#}aa^{\#}.$$

In addition,

$$ab^{\#}aa^{\#}=aaa^{\#}b^{\#}aa^{\#}=abb^{\#}a^{\#}aa^{\#}=aaa^{\#}b^{\#}=ab^{\#},$$

$$ba^{\#}bb^{\#}=b(bb^{\#}a^{\#})bb^{\#}=baa^{\#}b^{\#}bb^{\#}=baa^{\#}b^{\#}=bbb^{\#}a^{\#}=ba^{\#}.$$

Since $bb^{\#}ab=bb^{\#}aaa^{\#}b=aa^{\#}bbb^{\#}a=aa^{\#}ba=bb^{\#}aa$, $b^{\#}ab=b^{\#}aa$. Similarly, $a^{\#}ba=a^{\#}bb$. Since $bb^{\#}a^{\#}b^{\#}=bb^{\#}a^{\#}aa^{\#}b^{\#}=aa^{\#}b^{\#}bb^{\#}a^{\#}=aa^{\#}b^{\#}a^{\#}=bb^{\#}a^{\#}a^{\#}$, $ba^{\#}b^{\#}=ba^{\#}a^{\#}$. Hence, $b^{\#}a^{\#}b^{\#}=b^{\#}a^{\#}a^{\#}$. Similarly, $ab^{\#}a^{\#}=ab^{\#}b^{\#}$ and $a^{\#}b^{\#}a^{\#}=a^{\#}b^{\#}b^{\#}$. To sum up, we have the following equalities:

$$bb^{\#}a=aa^{\#}b, \tag{5.16}$$

$$a^{\#}b^{\#}b^{\#}=a^{\#}b^{\#}a^{\#}, \tag{5.17}$$

$$bb^{\#}a^{\#}=aa^{\#}b^{\#},\ aa^{\#}bb^{\#}=bb^{\#}aa^{\#}, \tag{5.18}$$

$$ab^{\#}aa^{\#}=ab^{\#},\ ba^{\#}bb^{\#}=ba^{\#}, \tag{5.19}$$

$$ab^{\#}a^{\#}=ab^{\#}b^{\#},\ ba^{\#}b^{\#}=ba^{\#}a^{\#}, \tag{5.20}$$

$$a^{\#}ba=a^{\#}bb,\ b^{\#}ab=b^{\#}aa. \tag{5.21}$$

(1) Let $x = a^\# + b^\# - \frac{1}{2}bb^\# a^\# - \frac{1}{2}a^\# bb^\# - \frac{1}{2}b^\# aa^\#$. By equalities (5.16), (5.18), and (5.19), we obtain that

$$\begin{aligned}
&(a+b)x = (a+b)(a^\# + b^\# - \frac{1}{2}bb^\# a^\# - \frac{1}{2}a^\# bb^\# - \frac{1}{2}b^\# aa^\#)\\
=&aa^\# + ab^\# - \frac{1}{2}ab^\# - \frac{1}{2}aa^\# bb^\# - \frac{1}{2}ab^\#\\
&+ ba^\# + bb^\# - \frac{1}{2}ba^\# - \frac{1}{2}ba^\# - \frac{1}{2}aa^\# bb^\#\\
=&aa^\# + bb^\# - aa^\# bb^\#
\end{aligned}$$

and

$$\begin{aligned}
&x(a+b) = (a^\# + b^\# - \frac{1}{2}bb^\# a^\# - \frac{1}{2}a^\# bb^\# - \frac{1}{2}b^\# aa^\#)(a+b)\\
=&a^\# a + b^\# a - \frac{1}{2}aa^\# bb^\# - \frac{1}{2}a^\# b - \frac{1}{2}b^\# a + a^\# b\\
&+ b^\# b - \frac{1}{2}aa^\# b^\# b - \frac{1}{2}a^\# b - \frac{1}{2}b^\# a\\
=&aa^\# + bb^\# - aa^\# bb^\#.
\end{aligned}$$

So $(a+b)x = x(a+b)$. By equalities (5.16) and (5.18), we have

$$\begin{aligned}
&x(a+b)x = (aa^\# + bb^\# - aa^\# bb^\#)(a^\# + b^\# - \frac{1}{2}bb^\# a^\# - \frac{1}{2}a^\# bb^\# - \frac{1}{2}b^\# aa^\#)\\
=&a^\# + aa^\# b^\# - \frac{1}{2}aa^\# b^\# - \frac{1}{2}a^\# bb^\# - \frac{1}{2}bb^\# a^\# + bb^\# a^\# + b^\# - \frac{1}{2}bb^\# a^\#\\
&- \frac{1}{2}aa^\# b^\# - \frac{1}{2}b^\# aa^\# - aa^\# b^\# - aa^\# b^\# + \frac{1}{2}aa^\# b^\# + \frac{1}{2}aa^\# b^\# + \frac{1}{2}bb^\# a^\#\\
=&a^\# + b^\# - \frac{1}{2}bb^\# a^\# - \frac{1}{2}a^\# bb^\# - \frac{1}{2}b^\# aa^\# = x,
\end{aligned}$$

and

$$\begin{aligned}
(a+b)x(a+b) &= (aa^\# + bb^\# - aa^\# bb^\#)(a+b)\\
&= a + bb^\# a - aa^\# bb^\# a + aa^\# b + b - aa^\# b = a + b.
\end{aligned}$$

So $(a+b)^\# = a^\# + b^\# - \frac{1}{2}bb^\# a^\# - \frac{1}{2}a^\# bb^\# - \frac{1}{2}b^\# aa^\#$.

(2) Let $y = a^\# - b^\# - a^\# a^\# b + b^\# b^\# a$. By equalities (5.19)-(5.21), then

$$\begin{aligned}
&(a-b)y = (a-b)(a^\# - b^\# - a^\# a^\# b + b^\# b^\# a)\\
&= aa^\# - ab^\# - a^\# b + ab^\# b^\# a - ba^\# + bb^\# + ba^\# a^\# b - b^\# a\\
&= aa^\# - ab^\# - a^\# b + ab^\# a^\# a - ba^\# + bb^\# + ba^\# b^\# b - b^\# a\\
&= aa^\# - a^\# b - b^\# a + bb^\#
\end{aligned}$$

and

$$\begin{aligned}y(a-b) &= (a^\# - b^\# - a^\# a^\# b + b^\# b^\# a)(a-b)\\ &= a^\# a - b^\# a - a^\# a^\# ba + b^\# b^\# aa - a^\# b + b^\# b + a^\# a^\# bb - b^\# b^\# ab\\ &= a^\# a - a^\# b - b^\# a + bb^\#.\end{aligned}$$

Hence, $(a-b)y = y(a-b)$. Since $a^\# b^\# a = a^\# aa^\# b^\# a = a^\# bb^\# a^\# a$, by equalities (5.16), (5.17), (5.18) and (5.20), we obtain

$$\begin{aligned}&y(a-b)y = (a^\# a - a^\# b - b^\# a + bb^\#)(a^\# - b^\# - a^\# a^\# b + b^\# b^\# a)\\ =&a^\# - aa^\# b^\# - a^\# a^\# b + aa^\# b^\# b^\# a - a^\# ba^\# + a^\# bb^\# + a^\# ba^\# a^\# b - a^\# b^\# a\\ &- b^\# aa^\# + b^\# ab^\# + b^\# a^\# b - b^\# ab^\# b^\# a + bb^\# a^\# - b^\# - bb^\# a^\# a^\# b + b^\# b^\# a\\ =&a^\# - aa^\# b^\# - a^\# a^\# b + aa^\# b^\# a^\# a - a^\# aa^\# ba^\# + a^\# bb^\# + a^\# aa^\# ba^\# a^\# b - a^\# b^\# a\\ &- b^\# aa^\# + b^\# bb^\# ab^\# + b^\# bb^\# a^\# b - b^\# bb^\# ab^\# b^\# a + bb^\# a^\# - b^\# - bb^\# a^\# b^\# b + b^\# b^\# a\\ =&a^\# - aa^\# b^\# - a^\# a^\# b + aa^\# b^\# - a^\# bb^\# a^\# a + a^\# bb^\# + a^\# bb^\# aa^\# a^\# b - a^\# bb^\# aa^\#\\ &- b^\# aa^\# + b^\# aa^\# bb^\# + b^\# aa^\# bb^\# - b^\# aa^\# bb^\# b^\# a + bb^\# a^\# - b^\# - bb^\# a^\# + b^\# b^\# a\\ =&a^\# - bb^\# a^\# - a^\# a^\# b + bb^\# a^\# - a^\# bb^\# + a^\# bb^\# + a^\# aa^\# b^\# b - a^\# bb^\# - b^\# aa^\#\\ &+ b^\# aa^\# + b^\# aa^\# - b^\# bb^\# a^\# a + bb^\# a^\# - b^\# - bb^\# a^\# + b^\# b^\# a\\ =&a^\# - b^\# - a^\# a^\# b + b^\# b^\# a = y.\end{aligned}$$

By equalities (5.16) and (5.21), we have

$$\begin{aligned}(a-b)y(a-b) &= (a^\# a - a^\# b - b^\# a + bb^\#)(a-b)\\ &= a - a^\# ba - b^\# aa + bb^\# a - aa^\# b + a^\# bb + b^\# ab - b\\ &= a - a^\# bb - b^\# aa + bb^\# a - bb^\# a + a^\# bb + b^\# aa - b = a - b.\end{aligned}$$

Thus $(a-b)^\# = a^\# - b^\# - a^\# a^\# b + b^\# b^\# a$.

According to the above two theorems, we have the following corollary.

Corollary 5.3([52, Corollary 3.3]) Let $a,\ b \in R^\#$ and $2 \in R^{-1}$. If $abb^\# = baa^\#$ and $bb^\# a = aa^\# b$, then

(1) $a + b \in R^\#$ and $(a+b)^\# = a^\# + b^\# - \frac{3}{2}a^\# bb^\#$.

(2) $a - b \in R^\#$ and $(a-b)^\# = a^\# - b^\#$.

Proof By the proof of the above two theorems, we have

$$
\begin{aligned}
&a^{\#}bb^{\#} = b^{\#}aa^{\#},\ bb^{\#}a^{\#} = aa^{\#}b^{\#},\\
&b^{\#}b^{\#}a = b^{\#}a^{\#}b = a^{\#}bb^{\#},\\
&ab = aaa^{\#}b = abb^{\#}a = baa^{\#}a = ba,\\
&a^{\#}a^{\#}b = a^{\#}b^{\#}a = b^{\#}a^{\#}a = a^{\#}bb^{\#},\\
&ab^{\#} = aaa^{\#}b^{\#} = abb^{\#}a^{\#} = baa^{\#}a^{\#} = ba^{\#},\\
&a^{\#}b = a^{\#}aa^{\#}b = a^{\#}bb^{\#}a = b^{\#}aa^{\#}a = b^{\#}a,\\
&aa^{\#}b^{\#} = a^{\#}(ab^{\#}) = (a^{\#}b)a^{\#} = b^{\#}aa^{\#},\\
&ba^{\#}a^{\#} = ab^{\#}a^{\#} = ab^{\#}aa^{\#}b^{\#} = aa^{\#}b^{\#} = a^2a^{\#}(a^{\#}bb^{\#})b^{\#}\\
&= a^2(a^{\#}b^{\#}a)a^{\#}b^{\#} = a^2a^{\#}a^{\#}(ba^{\#})b^{\#} = aa^{\#}(ab^{\#})b^{\#} = ab^{\#}b^{\#}.
\end{aligned}
$$

From Theorem 5.1, we obtain

$$
\begin{aligned}
(a+b)^{\#} &= a^{\#} + b^{\#} - \frac{1}{2}a^{\#}bb^{\#} - \frac{1}{2}bb^{\#}a^{\#} - \frac{1}{2}aa^{\#}b^{\#}\\
&= a^{\#} + b^{\#} - \frac{1}{2}a^{\#}bb^{\#} - aa^{\#}b^{\#}\\
&= a^{\#} + b^{\#} - \frac{1}{2}a^{\#}bb^{\#} - b^{\#}aa^{\#}\\
&= a^{\#} + b^{\#} - \frac{1}{2}a^{\#}bb^{\#} - a^{\#}bb^{\#}\\
&= a^{\#} + b^{\#} - \frac{3}{2}a^{\#}bb^{\#}
\end{aligned}
$$

and

$$
\begin{aligned}
(a-b)^{\#} &= a^{\#} - b^{\#} - ba^{\#}a^{\#} + ab^{\#}b^{\#}\\
&= a^{\#} - b^{\#}.
\end{aligned}
$$

6 The Group Inverse of a Product

Mary and Patrício [34] characterized the existence of the group inverse of a product of two regular elements by an unit and gave the corresponding expression. Let a, b be regular elements in R, with reflexive inverses a^+, b^+, respectively. Let also

$$w = \left(1 - bb^+\right)\left(1 - a^+a\right)$$